

POROSITIES OF MANDELBROT PERCOLATION

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ABSTRACT. We study porosities in the Mandelbrot percolation process. We show that, almost surely at almost all points with respect to the natural measure, the mean porosities of the set and the natural measure exist and are equal to each other for all parameter values outside of a countable exceptional set. As a corollary, we obtain that, almost surely at almost all points, the lower porosities of the set and the natural measure are equal to zero, whereas the upper porosities obtain their maximum values.

1. INTRODUCTION

The porosity of a set describes the sizes of holes in the set. The concept dates back to the 1920's when Denjoy introduced a notion which he called index (see [7]). In today's terminology, this index is called the upper porosity (see Definition 3.1). The term porosity was introduced by Dolženko in [8]. Intuitively, if the upper porosity of a set equals α , then, in the set, there are holes of relative size α at arbitrarily small distances. On the other hand, the lower porosity (see Definition 3.1) guarantees the existence of holes of certain relative size at all sufficiently small distances. The upper porosity turned out to be useful in order to describe properties of exceptional sets, for example, for measuring sizes of sets where certain functions are non-differentiable. For more details about the upper porosity, we refer to an article of Zajíček [35]. Mattila [26] utilised the lower porosity to find upper bounds for Hausdorff dimensions of set, and Salli [31] verified the corresponding results for packing and box counting dimensions.

It turns out that upper porosity cannot be used to estimate the dimension of a set (see [27, Section 4.12]). An observation that there are

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sets which are not lower porous but nevertheless contain so many holes that their dimension is smaller than the dimension of the ambient space, leads to the concept of mean porosity of a set introduced by Koskela and Rohde [23] in order to study the boundary behaviour of conformal and quasiconformal mappings. Mean porosity guarantees that certain percentage of scales, that is, distances which are integer powers of some fixed number, contain holes of fixed relative size. Koskela and Rohde showed that, if a subset of the m -dimensional Euclidean space is mean porous, then its Hausdorff and packing dimensions are smaller than m . For a modification of their definition, see Definition 3.6.

The lower porosity of a measure was introduced by Eckmann and E. and M. Järvenpää in [10], the upper one by Mera and Morán in [29] and the mean porosity by Beliaev and Smirnov in [2]. The relations between porosities and dimensions of sets and measures have been investigated, for example, in [2, 3, 14–17, 19, 33]. For further information on this subject, we refer to a survey by Shmerkin [32]. Porosity has also been used for studying the conical densities of measures (see [20, 21]).

Note that sets with same dimension may have different porosities. In [18], E. and M. Järvenpää and Mauldin and, in [34], Urbański characterised deterministic iterated function systems whose attractors have positive porosity. Porosities of random recursive constructions were studied in [18]. Particularly interesting random constructions are those in which the copies of the seed set are glued together in such a way that there are no holes left. Thus, the corresponding deterministic system would be non-porous and the essential question is whether the randomness in the construction makes the set or measure porous. A classical example is the Mandelbrot percolation process (also known as the fractal percolation) introduced by Mandelbrot in 1974 in [25] (see Section 2). In [18], it was shown that, almost surely, the points with minimum porosity as well as those with maximum porosity are dense in the limit set. However, the question about porosity of typical points and that of the natural measure remained open. Later, it turned out [6] that, for typical points, the lower porosity equals 0 and the upper one is equal to $\frac{1}{2}$ as conjectured in [18]. Indeed, this is a corollary of the results of Chen et al. in [6] dealing with estimates on the dimensions of sets of exceptional points regarding the porosity.

In this paper, we prove that the mean porosities of the natural measure and of the limit set exist and are equal to each other almost surely at almost all points with respect to the natural measure for all parameter values outside of a countable set (see Theorem 4.11). We also show that mean porosities are continuous as a function of parameter outside this exceptional set (see Theorem 4.5). Unlike the upper and lower porosities, the mean porosities of the set and the natural measure at typical points are non-trivial. Indeed, we prove that almost surely the mean porosities of the set and the natural measure are positive and

less than one for almost all points with respect to the natural measure for all non-trivial parameter values (see Corollaries 4.8 and 4.13). As an application of our results, we solve the conjecture of [18] completely (and give a new proof for the part solved in [6]) by showing that, almost surely for almost all points with respect to the natural measure, the lower porosities of the limit set and of the measure are equal to the minimum value of 0, the upper porosity of the set attains its maximum value of $\frac{1}{2}$ and the upper porosity of the measure also attains its maximum value of 1 (see Corollary 4.14).

The article is organised as follows. In Section 2, we explain some basic facts about the Mandelbrot percolation and, in Section 3, we define porosities and mean porosities and describe some of their properties. Finally, in Section 4, we prove our results about mean porosities of the limit set and of the natural measure in the Mandelbrot percolation process.

2. MANDELBROT PERCOLATION

We begin by recalling some basic facts about Mandelbrot percolation in the m -dimensional Euclidean space \mathbb{R}^m , where $m \in \mathbb{N} = \{1, 2, \dots\}$. Let $k \geq 2$ be an integer, $I = \{1, \dots, k^m\}$ and $I^* = \bigcup_{i=0}^{\infty} I^i$, where $I^0 = \emptyset$. An element $\sigma \in I^i$ is called a word and its length is $|\sigma| = i$. For all $\sigma \in I^i$ and $\sigma' \in I^j$, we denote by $\sigma * \sigma'$ the element of I^{i+j} whose first i coordinates are those of σ and the last j coordinates are those of σ' . For all $i \in \mathbb{N}$ and $\sigma \in I^* \cup I^{\mathbb{N}}$, denote by $\sigma|_i$ the word in I^i formed by the first i elements of σ . For $\sigma \in I^*$ and $\tau \in I^* \cup I^{\mathbb{N}}$, we write $\sigma \prec \tau$ if the sequence τ starts with σ .

Let Ω be the set of functions $\omega: I^* \rightarrow \{c, n\}$ equipped with the topology induced by the metric $\rho(\omega, \omega') = k^{-|\omega \wedge \omega'|}$, where

$$|\omega \wedge \omega'| = \min\{j \in \mathbb{N} \mid \exists \sigma \in I^j \text{ with } \omega(\sigma) \neq \omega'(\sigma)\}.$$

Each $\omega \in \Omega$ can be thought of as a code that tells us which cubes we choose (c) and which we neglect (n). More precisely, let $\omega \in \Omega$. We start with the unit cube $[0, 1]^m$ and denote it by J_{\emptyset} . We divide J_{\emptyset} into k^m closed k -adic cubes with side length k^{-1} , enumerate them with letters from alphabet I and repeat this procedure inside each subcube. For all $\sigma \in I^i$, we use the notation J_{σ} for the unique closed subcube of J_{\emptyset} with side length k^{-i} coded by σ . The image of $\eta \in I^{\mathbb{N}}$ under the natural projection from $I^{\mathbb{N}}$ to $[0, 1]^m$ is denoted by $x(\eta)$, that is,

$$x(\eta) = \bigcap_{i=0}^{\infty} J_{\eta|_i},$$

where $\eta|_0 = \emptyset$. If $\omega(\sigma) = n$ for $\sigma \in I^i$ then $J_\sigma(\omega) = \emptyset$, and if $\omega(\sigma) = c$ then $J_\sigma(\omega) = J_\sigma$. Define

$$K_\omega = \bigcap_{i=0}^{\infty} \bigcup_{\sigma \in I^i} J_\sigma(\omega).$$

Fix $0 \leq p \leq 1$. We make the above construction random by demanding that if J_σ is chosen then $J_{\sigma*j}$, $j = 1, \dots, k^m$, are chosen independently with probability p . Let P be the natural Borel probability measure on Ω , that is, for all $\sigma \in I^*$ and $j = 1, \dots, k^m$,

$$\begin{aligned} P(\omega(\emptyset) = c) &= 1, \\ P(\omega(\sigma*j) = c \mid \omega(\sigma) = c) &= p, \\ P(\omega(\sigma*j) = n \mid \omega(\sigma) = n) &= 1. \end{aligned}$$

It is a well-known result in the theory of branching processes that if the expectation of the number of chosen cubes of side length k^{-1} does not exceed one, then the limit set K_ω is P -almost surely an empty set (see [1, Theorem 1]). In our case, this expectation equals $k^m p$ and, thus, with positive probability, $K_\omega \neq \emptyset$ provided that $k^{-m} < p \leq 1$. According to [28, Theorem 1.1] (see also [22]), the Hausdorff dimension of K_ω is P -almost surely equal to

$$(2.1) \quad d = \frac{\log(k^m p)}{\log k} = m + \frac{\log p}{\log k}$$

provided that $K_\omega \neq \emptyset$. For P -almost all $\omega \in \Omega$, there exists a natural Radon measure ν_ω on K_ω (see [28, Theorem 3.2]) and, moreover, there is a natural Radon probability measure Q on $I^\mathbb{N} \times \Omega$ such that, for every Borel set $B \subset I^\mathbb{N} \times \Omega$, we have

$$(2.2) \quad Q(B) = \frac{1}{(\text{diam } J_\emptyset)^d} \int \mu_\omega(B_\omega) dP(\omega),$$

where $B_\omega = \{\eta \in I^\mathbb{N} \mid (\eta, \omega) \in B\}$, ν_ω is the image of μ_ω under the natural projection and $\text{diam } A$ is the diameter of a set A (see [12, (1.13)]).

We denote by $\text{card } A$ the number of elements in a set A . For a word $\sigma \in I^*$, consider the martingale $\{N_{j,\sigma} k^{-jd}\}_{j \in \mathbb{N}}$, where

$$N_{j,\sigma} = \text{card}\{\tau \in I^* \mid |\tau| = |\sigma| + j, \tau \succ \sigma \text{ and } \omega(\tau) = c\},$$

and denote its almost sure (finite) limit by $X_\sigma(\omega)$. For all $l \in \mathbb{N} \cup \{0\}$, define a random variable X_l on $I^\mathbb{N} \times \Omega$ by $X_l(\eta, \omega) = X_{\eta|_l}(\omega)$. It is easy to see that, for P -almost all $\omega \in \Omega$,

$$X_\sigma(\omega) = \sum_{\tau \in I^j} k^{-jd} X_{\sigma*\tau}(\omega) \mathbb{1}_{\{\omega(\sigma*\tau)=c\}}$$

for all $j \in \mathbb{N}$, where the characteristic function of a set A is denoted by $\mathbb{1}_A$. Further, for $\sigma, \tau \in I^*$, the random variables X_σ and X_τ are identically distributed (see [4, Proposition 1]) and, if $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, they

are independent. Thus, X_l , $l \in \mathbb{N} \cup \{0\}$, have the same distribution. According to [28, Theorem 3.2], the variables $X_\sigma(\omega)$ are related to the measure ν_ω for P -almost all $\omega \in \Omega$ by the formulae

$$(2.3) \quad \nu_\omega(J_\sigma) = (\text{diam } J_\sigma)^d X_\sigma(\omega) \text{ for all } \sigma \in I^* \text{ and}$$

$$(2.4) \quad \sum_{\substack{\tau \in I^j \\ J_\tau \cap B \neq \emptyset}} l_\tau^d X_\tau(\omega) \searrow \nu_\omega(B) \text{ as } j \rightarrow \infty \text{ for all Borel sets } B \subset K_\omega,$$

where $l_\tau = \text{diam } J_\tau = \text{diam } J_\emptyset k^{-j} \mathbb{1}_{\{\omega(\tau)=c\}}$.

By (2.2) and (2.3), expectations with respect to the measures P and Q are connected in the following way (see also [12, (1.16)]): if $j \in \mathbb{N}$ and $Y: I^{\mathbb{N}} \times \Omega \rightarrow \mathbb{R}$ is a random variable such that $Y(\eta, \omega) = Y(\eta', \omega)$ provided that $\eta|_j = \eta'|_j$, then

$$(2.5) \quad E_Q[Y] = E_P \left[\sum_{\substack{\sigma \in I^j \\ \omega(\sigma)=c}} k^{-jd} X_\sigma Y(\sigma, \cdot) \right].$$

Hence, we have

$$(2.6) \quad Q(X_l = 0) = 0 \text{ and } E_Q[X_l] = E_P[X_0^2] < \infty$$

for all $l \in \mathbb{N} \cup \{0\}$ (see [28, Theorem 2.1]).

3. POROSITIES

In this section, we define porosities and mean porosities of sets and measures and prove some basic properties for them.

Definition 3.1. Let $A \subset \mathbb{R}^m$, $x \in \mathbb{R}^m$ and $r > 0$. The local porosity of A at x at distance r is

$$\text{por}(A, x, r) = \sup\{\alpha \geq 0 \mid \text{there is } z \in \mathbb{R}^n \text{ such that } B(z, \alpha r) \subset B(x, r) \setminus A\},$$

where the open ball centred at x and with radius r is denoted by $B(x, r)$. The lower and upper porosities of A at x are defined as

$$\underline{\text{por}}(A, x) = \liminf_{r \rightarrow 0} \text{por}(A, x, r) \text{ and } \overline{\text{por}}(A, x) = \limsup_{r \rightarrow 0} \text{por}(A, x, r),$$

respectively. If $\underline{\text{por}}(A, x) = \overline{\text{por}}(A, x)$, the common value, denoted by $\text{por}(A, x)$, is called the porosity of A at x .

Definition 3.2. The lower and upper porosities of a Radon measure μ on \mathbb{R}^m at a point $x \in \mathbb{R}^m$ are defined by

$$\underline{\text{por}}(\mu, x) = \lim_{\varepsilon \rightarrow 0} \liminf_{r \rightarrow 0} \text{por}(\mu, x, r, \varepsilon) \text{ and } \overline{\text{por}}(\mu, x) = \lim_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow 0} \text{por}(\mu, x, r, \varepsilon),$$

respectively, where for all $r, \varepsilon > 0$,

$$\text{por}(\mu, x, r, \varepsilon) = \sup\{\alpha \geq 0 \mid \text{there is } z \in \mathbb{R}^m \text{ such that } B(z, \alpha r) \subset B(x, r) \text{ and } \mu(B(z, \alpha r)) \leq \varepsilon \mu(B(x, r))\}.$$

If the upper and lower porosities agree, the common value is called the porosity of μ at x and denoted by $\text{por}(\mu, x)$.

Remark 3.3. (a) In some sources, the condition $B(z, \alpha r) \subset B(x, r) \setminus A$ in Definition 3.1 is replaced by the condition $B(z, \alpha r) \cap A = \emptyset$, leading to the definition

$$\widetilde{\text{por}}(A, x, r) = \sup\{\alpha \geq 0 \mid \text{there is } z \in B(x, r) \text{ such that } B(z, \alpha r) \cap A = \emptyset\}.$$

It is not difficult to see that

$$\widetilde{\text{por}}(A, x) = \frac{\text{por}(A, x)}{1 - \text{por}(A, x)},$$

which is valid both for the lower and upper porosity. Indeed, this follows from two geometric observations. First, $B(z, \alpha r) \cap A = \emptyset$ with $z \in \partial B(x, r)$ if and only if $B(z, \alpha r) \subset B(x, (1+\alpha)r) \setminus A$ with $\partial B(z, \alpha r) \cap \partial B(x, (1+\alpha)r) \neq \emptyset$, where the boundary of a set B is denoted by ∂B . Second, at local minima and maxima of the function $r \mapsto \text{por}(A, x, r)$, we have $\partial B(z, \alpha r) \cap \partial B(x, (1+\alpha)r) \neq \emptyset$, and at local minima and maxima of the function $r \mapsto \widetilde{\text{por}}(A, x, r)$, we have $z \in \partial B(x, r)$.

(b) Unlike the dimension, the porosity is sensitive to the metric. For example, defining cube-porosities by using cubes instead of balls in the definition, there is no formula to convert porosities to cube-porosities or vice versa. It is easy to construct a set such that the cube-porosity attains its maximum value (at some point) but the porosity does not. Take, for example, the union of the x- and y-axes in the plane. However, the lower porosity is positive, if and only if the lower cube-porosity is positive.

(c) In general metric spaces, in addition to $B(z, \alpha r) \subset B(x, r) \setminus A$, it is sometimes useful to require that the empty ball $B(z, \alpha r)$ is inside the reference ball $B(x, r)$ also algebraically, that is, $d(x, z) + \alpha r \leq r$. For further discussion about this matter, see [30].

The lower and upper porosities give the relative sizes of the largest and smallest holes, respectively. Taking into considerations the frequency of scales where the holes appear, leads to the notion of mean porosity. We proceed by giving a definition which is adapted to the Mandelbrot percolation process. We will use the maximum metric ϱ , that is, $\varrho(x, y) = \max_{i \in \{1, \dots, m\}} \{|x_i - y_i|\}$, and denote by $B_\varrho(y, r)$ the open ball centred at y and with radius r with respect to this metric. Recall that the balls in the maximum metric are cubes whose faces are parallel to the coordinate planes.

Definition 3.4. Let $A \subset \mathbb{R}^m$, μ be a Radon measure on \mathbb{R}^m , $x \in \mathbb{R}^m$, $\alpha \in [0, 1]$ and $\varepsilon > 0$. For $j \in \mathbb{N}$, we say that A has an α -hole at scale j near x if there is a point $z \in Q_j^k(x)$ such that

$$B_\varrho(z, \tfrac{1}{2}\alpha k^{-j}) \subset Q_j^k(x) \setminus A.$$

Here $Q_j^k(x)$ is the half-open k -adic cube of side length k^{-j} containing x and $B_\varrho(z, \frac{1}{2}\alpha k^{-j})$ is called an α -hole. We say that μ has an (α, ε) -hole at scale j near x if there is a point $z \in Q_j^k(x)$ such that

$$B_\varrho(z, \frac{1}{2}\alpha k^{-j}) \subset Q_j^k(x) \text{ and } \mu(B_\varrho(z, \frac{1}{2}\alpha k^{-j})) \leq \varepsilon \mu(Q_j^k(x)).$$

Remark 3.5. Note that, unlike in Definition 3.1, we have divided the radius of the ball in the complement of A as well as that with small measure by 2 and, therefore, α may attain values between 0 and 1. The reason for this is that the point x may be arbitrarily close to the boundary of $Q_j^k(x)$ and, if the whole cube $Q_j^k(x)$ is empty, it is natural to say that there is a hole of relative size 1.

Definition 3.6. Let $\alpha \in [0, 1]$. The lower α -mean porosity of a set $A \subset \mathbb{R}^m$ at a point $x \in \mathbb{R}^m$ is

$$\underline{\kappa}(A, x, \alpha) = \liminf_{i \rightarrow \infty} \frac{N_i(A, x, \alpha)}{i}$$

and the upper α -mean porosity is

$$\overline{\kappa}(A, x, \alpha) = \limsup_{i \rightarrow \infty} \frac{N_i(A, x, \alpha)}{i},$$

where

$$N_i(A, x, \alpha) = \text{card}\{j \in \mathbb{N} \mid j \leq i \text{ and } A \text{ has an } \alpha\text{-hole at scale } j \text{ near } x\}.$$

In the case the limit exists, it is called the α -mean porosity and denoted by $\kappa(A, x, \alpha)$. The lower α -mean porosity of a Radon measure μ on \mathbb{R}^m at $x \in \mathbb{R}^m$ is

$$\underline{\kappa}(\mu, x, \alpha) = \lim_{\varepsilon \rightarrow 0} \liminf_{i \rightarrow \infty} \frac{\tilde{N}_i(\mu, x, \alpha, \varepsilon)}{i}$$

and the upper one is

$$\overline{\kappa}(\mu, x, \alpha) = \lim_{\varepsilon \rightarrow 0} \limsup_{i \rightarrow \infty} \frac{\tilde{N}_i(\mu, x, \alpha, \varepsilon)}{i},$$

where

$$\tilde{N}_i(\mu, x, \alpha, \varepsilon) = \text{card}\{j \in \mathbb{N} \mid j \leq i \text{ and } \mu \text{ has an } (\alpha, \varepsilon)\text{-hole at scale } j \text{ near } x\}.$$

If the lower and upper mean porosities coincide, the common value, denoted by $\kappa(\mu, x, \alpha)$, is called the α -mean porosity of μ .

Remark 3.7. Mean porosity is highly sensitive to the choice of parameters. The definition is given in terms of k -adic cubes. For the Mandelbrot percolation, this is natural. For general sets, fixing an integer $h > 1$, a natural choice is to say that A has an α -hole at scale j near x , if there is $z \in \mathbb{R}^m$ such that $B(z, \alpha h^{-j} r_0) \subset B(x, h^{-j} r_0) \setminus A$ for some (or for all) $h^{-1} < r_0 \leq 1$. However, the choice of r_0 and h

matters as will be shown in Example 3.8 below. Shmerkin proposed in [33] the following base and starting scale independent notion of lower mean porosity of a measure (which can be adapted for sets and upper porosity as well): a measure μ is lower (α, κ) -mean porous at a point $x \in \mathbb{R}^m$ if

$$\liminf_{\rho \rightarrow 1} (\log \frac{1}{\rho})^{-1} \int_{\rho}^1 \mathbb{1}_{\{r | \text{por}(\mu, x, r, \varepsilon) \geq \alpha\}} r^{-1} dr \geq \kappa \text{ for all } \varepsilon > 0.$$

The disadvantage of this definition is that it is more complicated to calculate than the discrete version. To avoid these problems, one option is to aim at qualitative results concerning all parameter values, as our approach will show.

Next we give a simple example demonstrating the dependence of mean porosity on the starting scale and the base of scales.

Example 3.8. Fix an integer $h > 1$. In this example, we use a modification of Definitions 3.4 and 3.6 where $A \subset \mathbb{R}^m$ has an α -hole at scale j near x , if there exists $z \in \mathbb{R}^m$ such that $B(z, \alpha h^{-j}) \subset B(x, h^{-j}) \setminus A$. Let $x \in \mathbb{R}^2$. We define a set $A \subset \mathbb{R}^2$ as follows. For all $i \in \mathbb{N} \cup \{0\}$, consider the half-open annulus $D(i) = \{y \in \mathbb{R}^2 \mid h^{-i-1} < |y - x| \leq h^{-i}\}$. Let $A = \bigcup_{i=0}^{\infty} D(3i+1) \cup D(3i+2)$, that is, we choose two annuli out of every three successive ones and leave the third one empty. In this case, $\kappa(A, x, \frac{1}{2}(1 - h^{-1})) = \frac{1}{3}$. If we replaced h by h^3 in the definition of scales, we would conclude that $\kappa(A, x, \frac{1}{2}(1 - h^{-1})) = 1$. (Note that the lower and upper porosities are equal.) If we define A by starting with the two filled annuli, that is, $A = \bigcup_{i=0}^{\infty} D(3i) \cup D(3i+1)$, then $\kappa(A, x, \frac{1}{2}(1 - h^{-1})) = \frac{1}{3}$ using scales determined by h and $\kappa(A, x, \frac{1}{2}(1 - h^{-1})) = 0$ if scales are determined by powers of h^3 . By mixing these construction in a suitable way, one easily finds an example where $\kappa(A, x, \frac{1}{2}(1 - h^{-1})) = \frac{1}{3}$ for scales given by h , but $\underline{\kappa}(A, x, \frac{1}{2}(1 - h^{-1})) = 0$ and $\bar{\kappa}(A, x, \frac{1}{2}(1 - h^{-1})) = 1$ if the scales are determined by h^3 .

We finish this section with some measurability results. For that we need some notation.

Definition 3.9. For all $j \in \mathbb{N}$ and $\alpha \in [0, 1]$, define a function $\chi_j^\alpha: I^{\mathbb{N}} \times \Omega \rightarrow \{0, 1\}$ by setting $\chi_j^\alpha(\eta, \omega) = 1$, if and only if K_ω has an α -hole at scale j near $x(\eta)$. Define a function $\bar{\chi}_j^\alpha: I^{\mathbb{N}} \times \Omega \rightarrow \{0, 1\}$ in the same way except that the α -hole is a closed ball instead of an open one. For all $\alpha \in (0, 1)$, $\varepsilon > 0$ and $j \in \mathbb{N}$, define a function $\chi_j^{\alpha, \varepsilon}: I^{\mathbb{N}} \times \Omega \rightarrow \{0, 1\}$ by setting $\chi_j^{\alpha, \varepsilon}(\eta, \omega) = 1$, if and only if ν_ω has an (α, ε) -hole at scale j near $x(\eta)$. Finally, define a function $\bar{\chi}_j^{\alpha, \varepsilon}: I^{\mathbb{N}} \times \Omega \rightarrow \{0, 1\}$ by setting $\bar{\chi}_j^{\alpha, \varepsilon}(\eta, \omega) = 1$, if and only if there exists $z \in Q_j^k(x(\eta))$ such that $\nu_\omega(\bar{B}_\varepsilon(z, \frac{1}{2}\alpha k^{-j})) < \varepsilon \nu_\omega(Q_j^k(x(\eta)))$. Here

the closed ball in metric ϱ centred at $z \in \mathbb{R}^m$ with radius $r > 0$ is denoted by $\overline{B}_\varrho(z, r)$.

Lemma 3.10. *The maps*

$$\begin{aligned} (\eta, \omega) &\mapsto \underline{\kappa}(K_\omega, x(\eta), \alpha), \\ (\eta, \omega) &\mapsto \overline{\kappa}(K_\omega, x(\eta), \alpha), \\ (\eta, \omega) &\mapsto \underline{\kappa}(\nu_\omega, x(\eta), \alpha) \text{ and} \\ (\eta, \omega) &\mapsto \overline{\kappa}(\nu_\omega, x(\eta), \alpha) \end{aligned}$$

are Borel measurable for all $\alpha \in [0, 1]$.

Proof. Note that $\overline{\chi}_j^\alpha(\cdot, \omega)$ is locally constant for all $\omega \in \Omega$, that is, its value depends only on $\eta|_j$. Further, suppose that $\overline{\chi}_j^\alpha(\eta, \omega) = 1$. Then K_ω has a closed α -hole H at scale j near $x(\eta)$. Since H and K_ω are closed, their distance is positive. So there exists a finite set $T \subset I^*$ such that $\omega(\tau) = n$ for all $\tau \in T$ and

$$H \subset \bigcup_{\tau \in T} J_\tau.$$

If $\omega' \in \Omega$ is close to ω , then $\omega'(\tau) = n$ for all $\tau \in T$, which implies that $\overline{\chi}_j^\alpha(\eta, \omega') = 1$. We conclude that $\overline{\chi}_j^\alpha$ is continuous at (η, ω) . Trivially, $\overline{\chi}_j^\alpha$ is lower semi-continuous at those points where $\overline{\chi}_j^\alpha(\eta, \omega) = 0$. Therefore, $\overline{\chi}_j^\alpha$ is lower semi-continuous.

Let α_i be a strictly increasing sequence approaching α . We claim that

$$(3.1) \quad \lim_{i \rightarrow \infty} \overline{\chi}_j^{\alpha_i}(\eta, \omega) = \chi_j^\alpha(\eta, \omega)$$

for all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$. Indeed, obviously $\chi_j^\alpha(\eta, \omega) \leq \overline{\chi}_j^{\alpha_i}(\eta, \omega)$ for all $i \in \mathbb{N}$, and the sequence $(\overline{\chi}_j^{\alpha_i}(\eta, \omega))_{i \in \mathbb{N}}$ is decreasing. Thus, it is enough to study the case $\lim_{i \rightarrow \infty} \overline{\chi}_j^{\alpha_i}(\eta, \omega) = 1$. Let $(\overline{B}_\varrho(z_i, \frac{1}{2}\alpha_i k^{-j}))_{i \in \mathbb{N}}$ be a corresponding sequence of closed holes. In this case, one may find a convergent subsequence of $(z_i)_{i \in \mathbb{N}}$ converging to $z \in \mathbb{R}^d$ and $B_\varrho(z, \frac{1}{2}\alpha k^{-j}) \subset Q_j^k(x(\eta)) \setminus K_\omega$, completing the proof of (3.1). As a limit of semi-continuous functions, χ_j^α is Borel measurable. Now $N_i(K_\omega, x(\eta), \alpha) = \sum_{j=1}^i \chi_j^\alpha(\eta, \omega)$, implying that the map $(\eta, \omega) \mapsto \underline{\kappa}(K_\omega, x(\eta), \alpha)$ (as well as the upper mean porosity) is Borel measurable.

By construction, the map $\omega \mapsto X_\tau(\omega)$ is Borel measurable for all $\tau \in I^*$. Therefore, $\omega \mapsto \nu_\omega(B)$ is a Borel map for all Borel sets $B \subset \mathbb{R}^m$ by (2.4). In particular, the map $(\eta, \omega) \mapsto \nu_\omega(\overline{B}_\varrho(z, \frac{1}{2}\alpha k^{-j})) - \varepsilon \nu_\omega(Q_j^k(x(\eta)))$ is Borel measurable for all $z \in \mathbb{R}^m$, $\alpha \in [0, 1]$, $\varepsilon > 0$ and $j \in \mathbb{N}$. Let $(z_i)_{i \in \mathbb{N}}$ be a dense set in $[0, 1]^m$. Let $s > 0$, $\alpha \in [0, 1]$ and $j \in \mathbb{N}$. Suppose that there exists $z \in Q_j^k(x(\eta))$ such that $\nu_\omega(\overline{B}_\varrho(z, \frac{1}{2}\alpha k^{-j})) < s$. Since the map $x \mapsto \nu_\omega(\overline{B}_\varrho(x, r))$ is upper semi-continuous, there exists $z_i \in Q_j^k(x(\eta))$ such that $\nu_\omega(\overline{B}_\varrho(z_i, \frac{1}{2}\alpha k^{-j})) < s$.

Thus, $\bar{\chi}_j^{\alpha,\varepsilon}$ is Borel measurable. Further, $\chi_j^{\alpha,\varepsilon}(\eta, \omega) = 1$ if and only if there exist an increasing sequence $(\alpha_i)_{i \in \mathbb{N}}$ tending to α and a decreasing sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ tending to ε such that $\bar{\chi}_j^{\alpha_i, \varepsilon_i}(\eta, \omega) = 1$. Therefore, $\chi_j^{\alpha,\varepsilon}$ is Borel measurable, and the claim follows as in the case of mean porosities of sets. \square

Remark 3.11. (a) Note that, for all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$, the function $\alpha \mapsto \chi_0^\alpha(\eta, \omega)$ is decreasing and, thus, the lower and upper mean porosity functions are also decreasing as functions of α .

(b) Later, we will need modifications of the functions χ_j^α defined in the proof of Lemma 3.10. Their Borel measurability can be proven analogously to that of χ_j^α .

4. RESULTS

In this section, we state and prove our results concerning mean porosities of Mandelbrot percolation and its natural measure. To prove the existence of mean porosity and to compare the mean porosities of the limit set and the construction measure, we need a tool to establish the validity of the strong law of large numbers for certain sequences of random variables. We will use [13, Theorem 1] (see also [24, Corollary 11]), which we state (in a simplified form) for the convenience of the reader.

Theorem 4.1. *Let $\{Y_n\}_{n \in \mathbb{N}}$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants $(\rho_k)_{k \in \mathbb{N}}$ such that*

$$\sup_{n \in \mathbb{N}} |\text{Cov}(Y_n, Y_{n+k})| \leq \rho_k$$

for all $k \in \mathbb{N}$. Assume that

$$\sum_{n=1}^{\infty} \frac{\text{Var}(Y_n) \log^2 n}{n^2} < \infty \text{ and } \sum_{k=1}^{\infty} \rho_k < \infty.$$

Then $\{Y_n\}_{n \in \mathbb{N}}$ satisfies the strong law of large numbers. Here the covariance and variance are denoted by Cov and Var , respectively.

We will apply Theorem 4.1 to stationary sequences of random variables which are indicator functions of events with equal probabilities. In this setup, all conditions of the theorem will be satisfied if

$$(4.1) \quad \sum_{j=1}^{\infty} \text{Cov}(Y_0, Y_j) < \infty$$

and, in particular, if Y_i is independent from Y_j once $|i - j|$ is greater than some fixed integer.

For all $\alpha \in [0, 1]$ and $j, r \in \mathbb{N}$, define $\chi_{j,r}^\alpha: I^{\mathbb{N}} \times \Omega \rightarrow \{0, 1\}$ similarly to χ_j^α with the exception that the whole hole is assumed to be in $J_{\eta|_j} \setminus J_{\eta|_{j+r}}$. Observe that $\chi_{j,r}^\alpha(\eta, \omega) = \chi_{j,r}^\alpha(\eta', \omega)$ provided that $\eta|_{j+r} = \eta'|_{j+r}$.

Therefore, for any $\tau \in I^{j+r}$, we may define $\chi_{j,r}^\alpha(\tau, \omega) = \chi_{j,r}^\alpha(\eta, \omega)$, where $\eta|_{j+r} = \tau$. Note that, given $\omega(\tau|_j) = c$, the value of the function $\chi_{j,r}^\alpha(\tau, \cdot)$ depends only on the restriction of ω to $J_{\tau|_j} \setminus J_{\tau|_{j+r}}$.

Lemma 4.2. *Let $\alpha \in [0, 1]$ and $r \in \mathbb{N}$. The random variables $\chi_{j,r}^\alpha$ and $\chi_{i,r}^\alpha$ are Q -independent for all $i, j \in \mathbb{N}$ with $|i - j| \geq r$ and*

$$\text{Cov}(\chi_{n,r}^\alpha, \chi_{n+k,r}^\alpha) = \text{Cov}(\chi_{0,r}^\alpha, \chi_{k,r}^\alpha)$$

for all $n, k, r \in \mathbb{N}$.

Proof. Observe that, for any $\tau \in I^{j+r}$, the variables X_τ and $\chi_{0,r}^\alpha(\tau, \cdot)$ are P -independent given $\omega(\tau) = c$. Recalling that $E_P[X_\tau \mid \omega(\tau) = c] = 1$ by (2.2) and (2.3), we obtain by (2.5) that

$$\begin{aligned} E_Q[\chi_{0,r}^\alpha] &= E_P\left[\sum_{\tau \in I^r} k^{-rd} \mathbb{1}_{\{\omega(\tau)=c\}} X_\tau \chi_{0,r}^\alpha(\tau, \cdot)\right] \\ &= E_P\left[\sum_{\tau \in I^r} k^{-rd} \mathbb{1}_{\{\omega(\tau)=c\}} E_P[\chi_{0,r}^\alpha(\tau, \cdot) \mid \omega(\tau) = c]\right]. \end{aligned}$$

Further, for any $j \in \mathbb{N}$ (using the above calculation in the third equality),

$$\begin{aligned} E_Q[\chi_{j,r}^\alpha] &= E_P\left[\sum_{\sigma \in I^j} k^{-jd} \mathbb{1}_{\{\omega(\sigma)=c\}} \sum_{\tau \in I^r} k^{-rd} \mathbb{1}_{\{\omega(\sigma*\tau)=c\}} X_{\sigma*\tau} \chi_{j,r}^\alpha(\sigma * \tau, \cdot)\right] \\ &= E_P\left[\sum_{\sigma \in I^j} k^{-jd} \mathbb{1}_{\{\omega(\sigma)=c\}} E_P\left[\sum_{\tau \in I^r} k^{-rd} \mathbb{1}_{\{\omega(\sigma*\tau)=c\}} X_{\sigma*\tau} \chi_{j,r}^\alpha(\sigma * \tau, \cdot) \mid \omega(\sigma) = c\right]\right] \\ &= E_P\left[\sum_{\sigma \in I^j} k^{-jd} \mathbb{1}_{\{\omega(\sigma)=c\}} E_Q[\chi_{0,r}^\alpha]\right] = E_Q[\chi_{0,r}^\alpha]. \end{aligned}$$

Let $i, j \in \mathbb{N}$ be such that $j - i \geq r$. Then (using the above calculation in the third and fourth equality)

$$\begin{aligned} E_Q[\chi_{i,r}^\alpha \chi_{j,r}^\alpha] &= E_P\left[\sum_{\tau \in I^{i+r}} k^{-(i+r)d} \mathbb{1}_{\{\omega(\tau)=c\}} \chi_{i,r}^\alpha(\tau, \cdot) \sum_{\sigma \in I^{j-i}} k^{-(j-i)d} \mathbb{1}_{\{\omega(\tau*\sigma)=c\}} X_{\tau*\sigma} \chi_{j,r}^\alpha(\tau * \sigma, \cdot)\right] \\ &= E_P\left[\sum_{\tau \in I^{i+r}} k^{-(i+r)d} \mathbb{1}_{\{\omega(\tau)=c\}} E_P[\chi_{i,r}^\alpha(\tau, \cdot) \mid \omega(\tau) = c] \right. \\ &\quad \times E_P\left[\sum_{\sigma \in I^{j-i}} k^{-(j-i)d} X_{\tau*\sigma} \mathbb{1}_{\{\omega(\tau*\sigma)=c\}} \chi_{j,r}^\alpha(\tau * \sigma, \cdot) \mid \omega(\tau) = c\right]\left. \right] \\ &= E_P\left[\sum_{\tau \in I^{i+r}} k^{-(i+r)d} \mathbb{1}_{\{\omega(\tau)=c\}} E_P[\chi_{i,r}^\alpha(\tau, \cdot) \mid \omega(\tau) = c] E_Q[\chi_{0,r}^\alpha] = E_Q[\chi_{i,r}^\alpha] E_Q[\chi_{j,r}^\alpha]. \right] \end{aligned}$$

The last claim follows from a similar calculation. \square

Next we prove a lemma which gives lower and upper bounds for mean porosities at typical points.

Lemma 4.3. *For all $\alpha \in (0, 1)$, we have*

$$E_Q[\bar{\chi}_0^\alpha] \leq \underline{\kappa}(K_\omega, x(\eta), \alpha) \leq \bar{\kappa}(K_\omega, x(\eta), \alpha) \leq E_Q[\chi_0^\alpha]$$

for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$.

Proof. Note that for every $\alpha \in (0, 1)$ and $r \in \mathbb{N}$ with $k^{-r} < \alpha$, we have

$$\chi_{j,r}^\alpha(\eta, \omega) \leq \chi_j^\alpha(\eta, \omega) \leq \chi_{j,r}^{\alpha-k^{-r}}(\eta, \omega)$$

for all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$ satisfying $x(\eta) \in K_\omega$. Recall that ν_ω is supported on K_ω for P -almost all $\omega \in \Omega$. Combining Lemma 4.2 and Theorem 4.1, we conclude that, for all $r \in \mathbb{N}$ and Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$, we have

$$\begin{aligned} E_Q[\chi_{0,r}^\alpha] &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \chi_{j,r}^\alpha(\eta, \omega) \leq \underline{\kappa}(K_\omega, x(\eta), \alpha) \\ &\leq \bar{\kappa}(K_\omega, x(\eta), \alpha) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \chi_{j,r}^{\alpha-k^{-r}}(\eta, \omega) = E_Q[\chi_{0,r}^{\alpha-k^{-r}}]. \end{aligned}$$

Observe that, for all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$ satisfying $x(\eta) \in K_\omega$, we have $\lim_{r \rightarrow \infty} \chi_{0,r}^\alpha(\eta, \omega) \geq \bar{\chi}_0^\alpha(\eta, \omega)$, since the distance between a closed α -hole and K_ω is positive. Further, the inequality $\chi_{0,r}^{\alpha-k^{-r}} \leq \bar{\chi}_0^{\alpha-2k^{-r}}$ is always valid and $\lim_{r \rightarrow \infty} \bar{\chi}_0^{\alpha-2k^{-r}} = \chi_0^\alpha$ by (3.1). Hence,

$$E_Q[\bar{\chi}_0^\alpha] \leq \underline{\kappa}(K_\omega, x(\eta), \alpha) \leq \bar{\kappa}(K_\omega, x(\eta), \alpha) \leq E_Q[\chi_0^\alpha]$$

for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$. \square

In fact, the upper bound we have found is an exact equality.

Proposition 4.4. *For all $\alpha \in (0, 1)$, we have that*

$$\bar{\kappa}(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^\alpha]$$

for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$.

Proof. We start by proving that, for all $\alpha \in (0, 1)$,

$$\lim_{j \rightarrow \infty} \text{Cov}(\chi_0^\alpha, \chi_j^\alpha) = 0.$$

Let $\alpha \in (0, 1)$. Note that, for all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$ satisfying $x(\eta) \in K_\omega$, we have

$$\chi_0^\alpha(\eta, \omega) \leq \chi_{0,r}^{\alpha-k^{-r}}(\eta, \omega) \leq \bar{\chi}_0^{\alpha-2k^{-r}}(\eta, \omega)$$

for all $r \in \mathbb{N}$ such that $2k^{-r} < \alpha$. Therefore, for all $j \in \mathbb{N}$ with $2k^{-j} < \alpha$, we have the following estimate

$$\text{Cov}(\chi_0^\alpha, \chi_j^\alpha) = E_Q[\chi_0^\alpha \chi_j^\alpha] - E_Q[\chi_0^\alpha] E_Q[\chi_j^\alpha] \leq E_Q[\chi_{0,j}^{\alpha-k^{-j}} \chi_j^\alpha] - E_Q[\chi_0^\alpha]^2.$$

Next we note that the random variables $\chi_{0,j}^{\alpha-k^{-j}}$ and χ_j^α are Q -independent (compare Lemma 4.2), hence

$$\text{Cov}(\chi_0^\alpha, \chi_j^\alpha) \leq E_Q[\chi_0^\alpha] E_Q[\chi_{0,j}^{\alpha-k^{-j}} - \chi_0^\alpha] \leq E_Q[\chi_0^\alpha] E_Q[\bar{\chi}_0^{\alpha-2k^{-j}} - \chi_0^\alpha].$$

Since $\lim_{j \rightarrow \infty} (\alpha - 2k^{-j}) = \alpha$, the equality (3.1) and the dominated convergence theorem imply that $\lim_{j \rightarrow \infty} E_Q[\bar{\chi}_0^{\alpha - 2k^{-j}} - \chi_0^\alpha] = 0$. Now, by Bernstein's theorem, the sequence $\frac{1}{n}N_n(A, x, \alpha) = \frac{1}{n} \sum_{i=1}^n \chi_i^\alpha$ converges in probability to $E_Q[\chi_0^\alpha]$. Once we have the convergence in probability, we can find a subsequence converging almost surely and, therefore, the upper bound in Theorem 4.3 is attained. \square

Define

$$D = \{\alpha \in (0, 1) \mid \beta \mapsto E_Q[\chi_0^\beta] \text{ is discontinuous at } \beta = \alpha\}.$$

Since $\beta \mapsto E_Q[\chi_0^\beta]$ is decreasing, the set D is countable.

Theorem 4.5. *For Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$, we have*

$$\kappa(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^\alpha]$$

for all $\alpha \in (0, 1) \setminus D$. In particular, for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$, the function $\alpha \mapsto \kappa(K_\omega, x(\eta), \alpha)$ is defined and continuous at all $\alpha \in (0, 1) \setminus D$.

Proof. Since $\chi_0^{\alpha'} \leq \bar{\chi}_0^\alpha \leq \chi_0^\alpha$ for all $\alpha' > \alpha$, we have that $E_Q[\bar{\chi}_0^\alpha] = E_Q[\chi_0^\alpha]$ for all $\alpha \in (0, 1) \setminus D$. Lemma 4.3 implies that, for all $\alpha \in (0, 1) \setminus D$, there exists a Borel set $B_\alpha \subset I^\mathbb{N} \times \Omega$ such that $\kappa(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^\alpha]$ for all $(\eta, \omega) \in B_\alpha$ and $Q(B_\alpha) = 1$. Let $(\alpha_i)_{i \in \mathbb{N}}$ be a dense set in $(0, 1)$. Since the functions $\alpha \mapsto \underline{\kappa}(K_\omega, x(\eta), \alpha)$ and $\alpha \mapsto \bar{\kappa}(K_\omega, x(\eta), \alpha)$ are decreasing, we have for all $(\eta, \omega) \in \bigcap_{i=1}^\infty B_{\alpha_i}$ that $\kappa(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^\alpha]$ for all $\alpha \in (0, 1) \setminus D$. Since $Q(\bigcap_{i=1}^\infty B_{\alpha_i}) = 1$, the proof is complete. \square

Proposition 4.6. *Suppose that $m = 2$ and $p > k^{-1}$. Then the set D is non-empty.*

Proof. Since $E_Q[\chi_0^{\alpha'}] \leq E_Q[\bar{\chi}_0^\alpha] \leq E_Q[\chi_0^\alpha]$ for all $\alpha' > \alpha$, it is enough to show that there exists $\alpha \in (0, 1)$ such that $E_Q[\bar{\chi}_0^\alpha] < E_Q[\chi_0^\alpha]$. This, in turn, follows if

$$(4.2) \quad Q(\{(\eta, \omega) \in I^\mathbb{N} \times \Omega \mid \bar{\chi}_0^\alpha(\eta, \omega) = 0 \text{ and } \chi_0^\alpha(\eta, \omega) = 1\}) > 0,$$

since $\bar{\chi}_0^\alpha \leq \chi_0^\alpha$.

It is shown in [11] that, if $m = 2$ and $p > k^{-1}$, the projection of K_ω onto the x -axis is the whole unit interval $[0, 1]$ with positive probability. In particular, K_ω intersects all the faces of $[0, 1]^2$ with positive probability. Let $\alpha = k^{-r}$ for some $r \in \mathbb{N}$. Fix $\sigma \in I^r$. Then there exists a Borel set $B \subset \Omega$ with $P(B) > 0$ such that, for all $\omega \in B$, we have $\omega(\sigma) = n$ and K_ω intersects all the faces of J_τ for all $\tau \in I^{r+1}$ with $\tau|_r \neq \sigma$. In this case, $\chi_0^\alpha(\eta, \omega) = 1$ and $\bar{\chi}_0^\alpha(\eta, \omega) = 0$ for all $\omega \in B$ for μ_ω -almost all $\eta \in B_\omega$. This implies inequality (4.2). \square

Remark 4.7. A similar construction as in the proof of Proposition 4.2 can be done for any positive $\alpha = \sum_{j=1}^n q_j k^{-r_j} < 1$, where $r_j \in \mathbb{N}$ and

$q_j \in \mathbb{Z}$, that is, for any hole which is a finite union of construction squares. We do not know whether $\kappa(K_\omega, x(\eta), \alpha)$ exists for $\alpha \in D$.

Corollary 4.8. *For P -almost all $\omega \in \Omega$ and for ν_ω -almost all $x \in K_\omega$, we have that*

$$0 < \underline{\kappa}(K_\omega, x, \alpha) \leq \overline{\kappa}(K_\omega, x, \alpha) < 1$$

for all $\alpha \in (0, 1)$, $\kappa(K_\omega, x, 0) = 1$ and $\kappa(K_\omega, x, 1) = 0$.

Proof. Since $0 < E_Q[\chi_0^\alpha] < 1$ for all $\alpha \in (0, 1)$ and the functions $\alpha \mapsto \underline{\kappa}(K_\omega, x(\eta), \alpha)$ and $\alpha \mapsto \overline{\kappa}(K_\omega, x(\eta), \alpha)$ are decreasing, the first claim follows from Theorem 4.5. The claim $\kappa(K_\omega, x, 0) = 1$ is obvious. Finally, if $\overline{\kappa}(K_\omega, x, 1) > 0$, K_ω has a 1-hole near x at scale j for some $j \in \mathbb{N}$. Hence, x should be on the boundary of the hole and $J_{\eta|_j}$ which, in turn, implies that K_ω has a 1-hole near x at all scales larger than j . Thus $\kappa(K_\omega, x, \alpha) = 1$ for all $\alpha \leq 1$ which is a contradiction with the first claim. \square

To study the mean porosities of the natural measure, we need some auxiliary results.

Proposition 4.9. *For all $s > 0$, the sequence $\{\mathbb{1}_{\{X_j \leq s\}}\}_{j \in \mathbb{N}}$ satisfies the strong law of large numbers.*

Proof. Since the sequence $(X_j)_{j \in \mathbb{N}}$ is stationary, we only have to check that the series (4.1) converges with $Y_j = \mathbb{1}_{\{X_j \leq s\}}$. Since X_j and $X_0 - k^{-jd}X_j$ are Q -independent (compare Lemma 4.2 or see the remark before [5, Lemma 10]), recalling that X_j and X_0 have the same distribution, we can make the following estimate

$$\begin{aligned} & \text{Cov}(\mathbb{1}_{\{X_0 \leq s\}}, \mathbb{1}_{\{X_j \leq s\}}) \\ &= Q(X_0 \leq s \text{ and } X_j \leq s) - Q(X_0 \leq s)Q(X_j \leq s) \\ &\leq Q(X_0 - k^{-jd}X_j \leq s \text{ and } X_j \leq s) - Q(X_0 \leq s)Q(X_j \leq s) \\ &= Q(X_j \leq s)(Q(X_0 - k^{-jd}X_j \leq s) - Q(X_0 \leq s)) \\ &= Q(X_0 \leq s)Q(s < X_0 \leq s + k^{-jd}X_j) \\ &\leq Q(X_0 \leq s)(Q(s < X_0 \leq s + k^{-\frac{1}{2}jd}) + Q(X_0 > k^{\frac{1}{2}jd})). \end{aligned}$$

By a result of Dubuc and Seneta [9] (see also [1, Theorem II.5.2]), the distribution of X_0 has a continuous P -density $q(x)$ on $(0, +\infty)$. From formula (2.5), we obtain

$$\begin{aligned} Q(s < X_0 \leq s + k^{-\frac{1}{2}jd}) &= E_Q[\mathbb{1}_{\{s < X_0 \leq s + k^{-\frac{1}{2}jd}\}}] \\ &= E_P[X_0 \mathbb{1}_{\{s < X_0 \leq s + k^{-\frac{1}{2}jd}\}}] \\ &\leq (s + k^{-\frac{1}{2}jd})P(s < X_0 \leq s + k^{-\frac{1}{2}jd}) \\ &\leq (s + k^{-\frac{1}{2}jd})k^{-\frac{1}{2}jd} \max_{x \in [s, s + k^{-\frac{1}{2}jd}]} q(x). \end{aligned}$$

Therefore, by Markov's inequality,

$$\sum_{j=1}^{\infty} \text{Cov}(\mathbb{1}_{\{X_0 \leq s\}}, \mathbb{1}_{\{X_j \leq s\}}) \leq$$

$$Q(X_0 \leq s) \left((s + k^{-\frac{1}{2}d}) \max_{x \in [s, s+k^{-\frac{1}{2}d}]} q(x) + E_Q(X_0) \right) \sum_{j=1}^{\infty} k^{-\frac{1}{2}jd} < \infty.$$

□

For all $\alpha \in (0, 1)$, $\varepsilon, \delta > 0$ and $j \in \mathbb{N}$, define a function $H_j^{\alpha, \varepsilon, \delta}: I^{\mathbb{N}} \times \Omega \rightarrow \{0, 1\}$ by setting $H_j^{\alpha, \varepsilon, \delta}(\eta, \omega) = 1$, if and only if ν_{ω} has an (α, ε) -hole at scale j near $x(\eta)$ but K_{ω} does not have an $(\alpha - \delta)$ -hole at scale j near $x(\eta)$.

Lemma 4.10. *Let $\alpha \in (0, 1)$. For all $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, for Q -almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$, we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n H_j^{\alpha, \varepsilon, \delta}(\eta, \omega) \leq \delta$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Proof. Fix $0 < \delta < \alpha$. Let $r \in \mathbb{N}$ be the smallest integer such that $2k^{-r} < \delta$. Let $\varepsilon > 0$. Assume that $H_j^{\alpha, \varepsilon, \delta}(\eta, \omega) = 1$ and denote by H the (α, ε) -hole at scale j near $x(\eta)$. Considering the relative positions of H and $J_{\eta|_{j+r}}$, we will argue that we arrive at the following possibilities:

- (i) If $J_{\eta|_{j+r}} \subset H$, we have $\nu_{\omega}(J_{\eta|_{j+r}}) \leq \varepsilon \nu_{\omega}(J_{\eta|_j})$.
- (ii) In the case $J_{\eta|_{j+r}} \not\subset H$, there exists $\tau_j \in I^{j+r}$ such that $\tau_j \neq \eta|_{j+r}$, $J_{\tau_j} \subset H$, $K_{\omega} \cap J_{\tau_j} \neq \emptyset$ and $\nu_{\omega}(J_{\tau_j}) \leq \varepsilon \nu_{\omega}(J_{\eta|_j})$.

Suppose that (i) is not valid. Since $H_j^{\alpha, \varepsilon, \delta}(\eta, \omega) = 1$, the set K_{ω} does not have an $(\alpha - \delta)$ -hole at scale j near $x(\eta)$. Observe that

$$H \setminus \bigcup_{\substack{\sigma \in I^{j+r} \\ J_{\sigma} \not\subset H}} J_{\sigma}$$

contains a cube with side length $(\alpha - \delta)k^{-j}$ since $2k^{-r} < \delta$. Since $J_{\eta|_{j+r}} \not\subset H$, there exists $\tau_j \in I^{j+r}$ as in (ii).

Next we estimate how often (i) or (ii) may happen for Q -typical $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$. We start by considering the case (i). We denote by $A_j^{1, \varepsilon}$ the event that $\nu_{\omega}(J_{\eta|_{j+r}}) \leq \varepsilon \nu_{\omega}(J_{\eta|_j})$, that is, according to (2.3),

$$A_j^{1, \varepsilon} = \{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid$$

$$X_{\eta|_{j+r}}(\omega) \leq \frac{\varepsilon}{1 - \varepsilon} \sum_{\substack{\tau \in I^{j+r} \\ \eta|_j \prec \tau, \tau \neq \eta|_{j+r}, \omega(\tau) = c}} X_{\tau}(\omega)\}.$$

For all $s > 0$, let

$$A_{j,1}^{s,\varepsilon} = \{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid X_{\eta|_{j+r}}(\omega) \leq \frac{\varepsilon s}{1-\varepsilon}\}$$

and $A_{j,2}^s = \{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid \sum_{\substack{\tau \in I^{j+r} \\ \eta|_j \prec \tau, \tau \neq \eta|_{j+r}, \omega(\tau)=c}} X_{\tau}(\omega) > s\}.$

In the case (ii), let

$$A_j^{2,\varepsilon} = \{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid \exists \tau \in I^{j+r} \text{ such that } \tau \succ \eta|_j, \\ \tau \neq \eta|_{j+r} \text{ and } 0 < k^{-rd} X_{\tau}(\omega) \leq \varepsilon X_{\eta|_j}(\omega)\}.$$

Recall that, for any $\tau \in I^*$, we have $P(X_{\tau}(\omega) > 0 \mid K_{\omega} \cap J_{\tau} \neq \emptyset) = 1$ by [28, Theorem 3.4]. Defining

$$A_{j,3}^s = \{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid X_{\eta|_j}(\omega) > s\} \text{ and} \\ A_{j,4}^{s,\varepsilon} = \{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid \exists \tau \in I^{j+r} \text{ such that } \tau \succ \eta|_j, \\ \tau \neq \eta|_{j+r} \text{ and } 0 < k^{-rd} X_{\tau}(\omega) \leq \varepsilon s\},$$

we have

$$H_j^{\alpha,\varepsilon,\delta} \leq \mathbb{1}_{A_j^{1,\varepsilon}} + \mathbb{1}_{A_j^{2,\varepsilon}} \leq \mathbb{1}_{A_{j,1}^{s,\varepsilon}} + \mathbb{1}_{A_{j,2}^s} + \mathbb{1}_{A_{j,3}^s} + \mathbb{1}_{A_{j,4}^{s,\varepsilon}}.$$

By Proposition 4.9, the functions $\mathbb{1}_{A_{j,1}^{s,\varepsilon}}$ and $\mathbb{1}_{A_{j,3}^s}$ satisfy the strong law of large numbers. The same is true for $\mathbb{1}_{A_{j,2}^s}$ and $\mathbb{1}_{A_{j,4}^{s,\varepsilon}}$ by Theorem 4.1, since $A_{j,2}^s$ and $A_{i,2}^s$ as well as $A_{j,4}^{s,\varepsilon}$ and $A_{i,4}^{s,\varepsilon}$ are Q -independent if $|i-j| \geq r$. This can be seen similarly as in the proof of Lemma 4.2. Hence, we obtain the estimate

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n H_j^{\alpha,\varepsilon,\delta}(\eta, \omega) \leq Q(A_{0,1}^{s,\varepsilon}) + Q(A_{0,2}^s) + Q(A_{0,3}^s) + Q(A_{0,4}^{s,\varepsilon})$$

for Q -almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$. Observe that the left hand side of the above inequality decreases as ε decreases for all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$. For all large enough s , the value of $Q(A_{0,2}^s) + Q(A_{0,3}^s)$ is less than $\frac{1}{2}\delta$. Fix such an $0 < s < \infty$. According to (2.6), we have $Q(X_r = 0) = 0$. Therefore, for all ε small enough, we have $Q(A_{0,1}^{s,\varepsilon}) + Q(A_{0,4}^{s,\varepsilon}) < \frac{1}{2}\delta$, completing the proof. \square

Now we are ready to prove that the mean porosity of the natural measure equals that of the Mandelbrot percolation set.

Theorem 4.11. *For Q -almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$, we have*

$$\kappa(K_{\omega}, x(\eta), \alpha) = \kappa(\nu_{\omega}, x(\eta), \alpha)$$

for all $\alpha \in (0, 1) \setminus D$.

Proof. For all $\alpha \in (0, 1)$, $\varepsilon > 0$ and $j \in \mathbb{N}$, let χ_j^α and $\chi_j^{\alpha, \varepsilon}$ be as in Definition 3.9 and $H_j^{\alpha, \varepsilon, \delta}$ as in Lemma 4.10. The inequalities

$$\underline{\kappa}(K_\omega, x(\eta), \alpha) \leq \underline{\kappa}(\nu_\omega, x(\eta), \alpha) \text{ and } \overline{\kappa}(K_\omega, x(\eta), \alpha) \leq \overline{\kappa}(\nu_\omega, x(\eta), \alpha)$$

are obvious for all $\alpha \in (0, 1)$ and $(\eta, \omega) \in I^\mathbb{N} \times \Omega$ since $\chi_j^\alpha \leq \chi_j^{\alpha, \varepsilon}$. Therefore, for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$, we have $\kappa(K_\omega, x(\eta), \alpha) \leq \underline{\kappa}(\nu_\omega, x(\eta), \alpha)$ for all $\alpha \in (0, 1) \setminus D$ by Theorem 4.5.

Let $0 < \delta < \alpha$ and $\varepsilon > 0$. Since $\chi_j^{\alpha, \varepsilon} \leq \chi_j^{\alpha - \delta} + H_j^{\alpha, \varepsilon, \delta}$ for all $j \in \mathbb{N}$, we have, by Lemma 4.10, for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$

$$\begin{aligned} \overline{\kappa}(\nu_\omega, x(\eta), \alpha) &= \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \chi_j^{\alpha, \varepsilon}(\eta, \omega) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \chi_j^{\alpha - \delta}(\eta, \omega) + \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n H_j^{\alpha, \varepsilon, \delta}(\eta, \omega) \\ &\leq \overline{\kappa}(K_\omega, x(\eta), \alpha - \delta) + \delta. \end{aligned}$$

Since $\alpha \mapsto \overline{\kappa}(K_\omega, x(\eta), \alpha)$ is continuous at all $\alpha \in (0, 1) \setminus D$ by Theorem 4.5, we conclude that, for all $\alpha \in (0, 1) \setminus D$, we have for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$ that $\overline{\kappa}(\nu_\omega, x(\eta), \alpha) \leq \kappa(K_\omega, x(\eta), \alpha)$. As in the proof of Theorem 4.5, we see that the order of the quantifiers may be reversed. \square

Before stating a corollary of the previous theorem, we prove a lemma, which is well known, but for which we did not find a reference.

Lemma 4.12. *Let V be a coordinate hyperplane and let e be the unit vector perpendicular to V . Then, for all $t \in [0, 1]$,*

$$P(\nu_\omega(K_\omega \cap (te + V)) > 0) = 0.$$

Proof. Fix $t \in [0, 1]$. According to (2.4),

$$\nu_\omega(K_\omega \cap (te + V)) = \lim_{j \rightarrow \infty} (\text{diam } J_\emptyset)^d \sum_{\substack{\tau \in I^j \\ J_\tau \cap (te + V) \neq \emptyset}} k^{-jd} X_\tau(\omega) \mathbb{1}_{\{\omega(\tau) = c\}},$$

and the above sequence decreases monotonically as j tends to infinity. Hence,

$$\begin{aligned} &E_P[\nu_\omega(K_\omega \cap (te + V))] \\ &\leq (\text{diam } J_\emptyset)^d \lim_{j \rightarrow \infty} E_P \left[\sum_{\substack{\tau \in I^j \\ J_\tau \cap (te + V) \neq \emptyset}} k^{-jd} X_\tau(\omega) \mathbb{1}_{\{\omega(\tau) = c\}} \right]. \end{aligned}$$

Note that, without the restriction $J_\tau \cap (te + V) \neq \emptyset$, the expectation on the right hand side equals 1. Since the restriction $J_\tau \cap (te + V) \neq \emptyset$ determines an exponentially decreasing proportion of indices as j tends to infinity and since the random variables $k^{-jd} X_\tau \mathbb{1}_{\{\omega(\tau) = c\}}$ have the same distribution, the limit of the expectation equals 0. \square

Next corollary is the counterpart of Corollary 4.8 for mean porosities of the natural measure.

Corollary 4.13. *For P -almost all $\omega \in \Omega$ and for ν_ω -almost all $x \in K_\omega$, we have*

$$0 < \underline{\kappa}(\nu_\omega, x, \alpha) \leq \overline{\kappa}(\nu_\omega, x, \alpha) < 1$$

for all $\alpha \in (0, 1)$, $\kappa(\nu_\omega, x, 0) = 1$ and $\kappa(\nu_\omega, x, 1) = 0$.

Proof. The first claim follows from Corollary 4.8, Theorem 4.11 and the monotonicity of the functions $\alpha \mapsto \underline{\kappa}(\nu_\omega, x, \alpha)$ and $\alpha \mapsto \overline{\kappa}(\nu_\omega, x, \alpha)$. Since $\underline{\kappa}(K_\omega, x, 0) \leq \underline{\kappa}(\nu_\omega, x, 0)$, the second claim follows from Corollary 4.8. Note that $\chi_j^{1,\epsilon}(\eta, \omega) = 1$ only if $\nu_\omega(\partial J_{\eta|_j}) > 0$. Therefore, the last claim follows from Lemma 4.12. \square

The following corollary solves completely Conjecture 3.2 stated in [18].

Corollary 4.14. *For P -almost all $\omega \in \Omega$ and for ν_ω -almost all $x \in K_\omega$, we have*

$$\underline{\text{por}}(K_\omega, x) = \underline{\text{por}}(\nu_\omega, x) = 0, \quad \overline{\text{por}}(K_\omega, x) = \frac{1}{2} \text{ and } \overline{\text{por}}(\nu_\omega, x) = 1.$$

Proof. By Corollary 4.8, for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$, we have that $\overline{\kappa}(K_\omega, x(\eta), \alpha) < 1$ for all $\alpha > 0$. Hence, for P -almost all $\omega \in \Omega$ and for ν_ω -almost all $x \in K_\omega$, there are, for all $\alpha > 0$, arbitrarily large $i \in \mathbb{N}$ such that K_ω does not have an α -hole at scale i near x which is contained in $Q_i^k(x)$. Note that, in Definitions 3.1 and 3.2, the holes are defined using balls while the mean porosities are defined in terms of k -adic cubes (see Definitions 3.4 and 3.6). Therefore, it is possible that $B(x, k^{-i})$ contains an α -hole which is outside the construction cube $Q_i^k(x)$ if x is close to the boundary of $Q_i^k(x)$. We show that there are infinitely many $i \in \mathbb{N}$ such that this will not happen.

Fix $\alpha \in (0, \frac{1}{4})$ and $r > 8$ large enough so that $2k^{-r} < \alpha$. Let $I' \subset I^r$ be the set of words such that, for all $\tau \in I'$, the ϱ -distance from all points of J_τ to the centre of J_\emptyset is at most $\frac{1}{4}$. For all $i \in \mathbb{N}$, define $Y_i^\alpha: I^\mathbb{N} \times \Omega \rightarrow \{0, 1\}$ by setting $Y_i^\alpha(\eta, \omega) = 1$, if and only if $J_{\eta|_i}$ is chosen, $\eta|_{i+r}$ ends with a word from I' and K_ω does not have an $\frac{1}{2}\alpha$ -hole at scale i near $x(\eta)$ which is completely inside $J_{\eta|_i} \setminus J_{\eta|_{i+r}}$. Note that if $x(\eta) \in K_\omega$ and K_ω has an α -hole at scale i near $x(\eta)$, then at least half of this hole is in $J_{\eta|_i} \setminus J_{\eta|_{i+r}}$. Thus, K_ω does not have an α -hole at scale i near $x(\eta)$ if $Y_i^\alpha(\eta, \omega) = 1$. Since for indices i and j with $|i - j| \geq r$, the events $\{(\eta, \omega) \in I^\mathbb{N} \times \Omega \mid Y_i^\alpha(\eta, \omega) = 1\}$ and $\{(\eta, \omega) \in I^\mathbb{N} \times \Omega \mid Y_j^\alpha(\eta, \omega) = 1\}$ are Q -independent (compare with Lemma 4.2), the averages of random variables $Y_i^\alpha(\eta, \omega)$ converge to $E_Q(Y_0^\alpha) > 0$ for Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$. If $Y_i^\alpha(\eta, \omega) = 1$, then $B(x(\eta), \frac{1}{4}k^{-i}) \subset J_{\eta|_i}$ and there is no $z \in B(x(\eta), \frac{1}{4}k^{-i})$ such that $B(z, \frac{1}{2}\alpha k^{-i}) \subset B(x(\eta), \frac{1}{4}k^{-i}) \setminus K_\omega$. Therefore, $\text{por}(K_\omega, x(\eta), \frac{1}{4}k^{-i}) \leq$

2α . A similar argument shows that $\text{por}(\nu_\omega, x(\eta), \frac{1}{4}k^{-i}) \leq 2\alpha$. Let $(\alpha_j)_{j \in \mathbb{N}}$ and $(\varepsilon_k)_{k \in \mathbb{N}}$ be sequences tending to 0. For Q -almost all $(\eta, \omega) \in I^\mathbb{N} \times \Omega$, we have for all $j, k \in \mathbb{N}$ that there are infinitely many scales $i \in \mathbb{N}$ such that

$$\text{por}(K_\omega, x(\eta), \frac{1}{4}k^{-i}) < \alpha_j \text{ and } \text{por}(\nu_\omega, x(\eta), \frac{1}{4}k^{-i}, \varepsilon_k) < \alpha_j.$$

Thus, we conclude that

$$\underline{\text{por}}(K_\omega, x) = 0 = \underline{\text{por}}(\nu_\omega, x)$$

for P -almost all $\omega \in \Omega$ and for ν_ω -almost all $x \in K_\omega$.

Since $\underline{\kappa}(K_\omega, x, \alpha) > 0$ and $\underline{\kappa}(\nu_\omega, x, \alpha) > 0$ for all $\alpha < 1$, we deduce that

$$\overline{\text{por}}(K_\omega, x) = \frac{1}{2} \text{ and } \overline{\text{por}}(\nu_\omega, x) = 1$$

for P -almost all $\omega \in \Omega$ and for ν_ω -almost all $x \in K_\omega$. \square

Remark 4.15. Theorems 4.5 and 4.11 should extend to homogeneous random self-similar sets satisfying the random strong open set condition.

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